MATH 210: Introduction to Analysis

Fall 2017-2018, Midterm 1, Duration: 60 min.

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Exercise	Points	Scores
1	20	
2	15	
3	23	
4	15	
5	27	
Total	100	

INSTRUCTIONS:

(a) Explain your answers precisely and clearly to ensure full credit.

(b) No book. No notes. No calculators.

Exercise 1. Let A be a nonempty bounded subset of $\mathbb{R}$. Let $\lambda \in \mathbb{R}$ and define

$$\lambda A = \{\lambda a \mid a \in A\}.$$

(a) **(5 points)** Prove that $\sup(\lambda A)$ exists.

- $A \neq \phi \Rightarrow \lambda A \neq \phi$
- A bdd $\Rightarrow$ There is $M > 0$, s.t. $|a| \leq M$ for all $a \in A$
  $\Rightarrow |\lambda a| \leq |\lambda| M$. $\forall$ for all $\lambda a \in A$. $\Rightarrow \lambda A$ bdd.

$\lambda A \neq \phi$ and bdd $\Rightarrow$ sup λA exists

(b) **(10 points)** Prove that $\sup(\lambda A) = \max\{\lambda \sup A, \lambda \inf A\}$.

- $\lambda = 0$ $\quad 0 = \sup \lambda A = \max\{\lambda \sup A, \lambda \inf A\} = 0$
- $\lambda > 0$ $\quad a \leq \sup A \Rightarrow \lambda a \leq \lambda \sup A$
  $\Rightarrow \sup \lambda A \leq \lambda \sup A$

Moreover $\quad \lambda a \leq \sup \lambda A \Rightarrow a \leq \frac{\sup \lambda A}{\lambda}$
$\Rightarrow \sup A \leq \frac{\sup \lambda A}{\lambda}$

Therefore $\sup \lambda A = \lambda \sup A$.

- $\lambda < 0$ $\quad \inf A \leq a \Rightarrow \lambda a \leq \lambda \inf A \Rightarrow \sup \lambda A \leq \lambda \inf A$

Moreover $\quad \lambda a \leq \sup \lambda A \Rightarrow a \geq \frac{\sup \lambda A}{\lambda} \Rightarrow \inf A \geq \frac{\sup \lambda A}{\lambda}$

Therefore $\sup \lambda A = \lambda \inf A$.

(c) **(5 points)** In case A is unbounded, may $\sup(\lambda A)$ exist? Yes

<u>example 1</u>: $A = \mathbb{R}$ $\quad \lambda = 0$. $\quad (\lambda A = \{0\})$.

<u>example 2</u>: $A = (1, \infty)$ $\quad \lambda = -1$
$(\lambda A = (-\infty, -1))$

Exercise 2.

(a) **(5 points)** Let $\{x_n\}$ be a sequence. Recall the definition of convergence of $\{x_n\}$.

See below

(b) **(10 points)** Prove using the definition of convergence that $\underbrace{\left\{\frac{2n^2+5n+\sqrt{n}}{n^2+3n}\right\}}_{x_n}$ converges to 2.

Let $\varepsilon > 0$

By Archimedean pty, there is n_0 s.t

$\frac{1}{n_0} < \varepsilon$. Now for $n \geq n_0$

Compute: $\left|\frac{2n^2+5n+\sqrt{n}}{n^2+3n} - 2\right| = \left|\frac{n-\sqrt{n}}{n^2+3n}\right|$

since ≥ 0

$$\stackrel{\text{since} \geq 0}{=} \frac{n-\sqrt{n}}{n^2+3n} < \frac{n}{n^2+3n}$$

$$< \frac{n}{n^2}$$

$$= \frac{1}{n} \leq \frac{1}{n_0} < \varepsilon$$

$\Rightarrow \quad x_n \to 2.$

Exercise 3. Let $\{x_n\}$ be a bounded sequence. Recall that

$$\overline{\lim} x_n = \sup\{x_k \mid k \geq n\}$$

and

$$\underline{\lim} x_n = \inf\{x_k \mid k \geq n\}$$

Define $y_n = \sup\{x_k \mid k \geq n\}$ and $z_n = \inf\{x_k \mid k \geq n\}$

(a) i. **(2 points)** Show that the sequence $\{z_n\}$ is bounded.

$\{x_n\}$ bdd $\Rightarrow$ There is $M > 0$ s.t. $|x_n| < M$ for all n.

$\Rightarrow |z_n| \leq M$ for all n $\Rightarrow \{z_n\}$ bdd.

ii. **(5 points)** Study the monotonicity of the sequence $\{z_n\}$.

$\{x_k \mid k \geq n+1\} \subset \{x_k \mid k \geq n\}$.

$$\Rightarrow \underbrace{\inf\{\text{———}\}}_{z_{n+1}} \geq \underbrace{\inf\{\text{———}\}}_{z_n}$$

$\Rightarrow \{z_n\} \nearrow$

iii. **(2 points)** Deduce that $\{z_n\}$ converges.

i. and ii $\Rightarrow \{z_n\}$ cv.

(b) Consider the sequence defined by $x_n = (-1)^n \left(1 + \frac{1}{n}\right)$.

i. **(5 points)** Find y_n.

$$\begin{cases} y_{2n} = 1 + \frac{1}{2n} \\ y_{2n+1} = 1 + \frac{1}{2n+2} \end{cases}$$

ii. (2 points) Deduce $\overline{\lim}\, x_n$.

$y_{2n} \to 1$, $y_{2n+1} \to 1$ By HW lec 5, this implies $y_n \to 1$.

iii. (5 points) Find z_n.

$$z_{2n} = -1 - \frac{1}{2n+1}$$

$$z_{2n+1} = -\frac{1}{[illegible]} - \frac{1}{2n+1}$$

iv. (2 points) Deduce $\underline{\lim}\, x_n$.

For same reason as (b) ii. $\underline{\lim}\, x_n = \lim z_n = -1$

Exercise 4.

(a) **(10 points)** Let $\{x_n\}$ be a monotonic sequence such that the subsequence $\{x_{2n}\}$ converges. Prove that $\{x_n\}$ converges.

Assume for instance $\{x_n\} \nearrow$.

Since $\{x_{2n}\}$ cv then $\{x_{2n}\}$ is bounded, that is, there is $M>0$ s.t. $|x_{2n}|<M$ for all n.

We have:

since $x_n \nearrow$

$$-M < x_{2n} \leq x_{2n+1} \leq x_{2n+2} < M$$

$\Rightarrow$ $|x_{2n+1}| < M$ for all n.

This proves $|x_n| < M$ for all n.

$\{x_n\}$ is bdd and $\nearrow$ $\Rightarrow$ $\{x_n\}$ cv.

(b) **(5 points)** Show that the previous conclusion may fail in case $\{x_n\}$ is not assumed monotonic.

$x_n = (-1)^n$.

Exercise 5.

(a) Let $0 \leq c < 1$. Let $\{x_n\}$ be a sequence satisfying $\left|\frac{x_{n+1}}{x_n}\right| < c$ for all n.

i. **(5 points)** Find a relation between x_n and x_0.

$$\left.\begin{aligned} |x_n| &< c\,|x_{n-1}| \\ |x_{n-1}| &< c\,|x_{n-2}| \\ &\vdots \\ |x_1| &< c\,|x_0| \end{aligned}\right\} |x_n| < c^n |x_0|$$

ii. **(2 points)** Deduce that x_n converges to 0.

Since $0 \leq c < 1$, $c^n \to 0$

$\underset{\downarrow 0}{0} \leq |x_n| < \underset{\downarrow 0}{c^n}\,|x_0|$. Sandwich $\Rightarrow x_n \to 0$.

iii. **(5 points)** Does the result still hold $c = 1$?

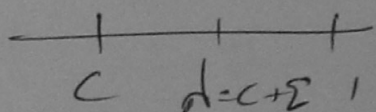
No, for instance $x_n = 1 + \frac{1}{n}$ is s.t

$\begin{cases} |x_{n+1}| < |x_n| \\ x_n \not\to 0. \end{cases}$

(b) Let $0 \leq c < 1$. Assume now that $\left|\frac{x_{n+1}}{x_n}\right|$ converges to c.

i. **(5 points)** Prove that there exist $0 < \lambda < 1$ and an integer n_0 such that if $n \geq n_0$ then $|x_n| \leq \lambda^{n-n_0}|x_{n_0}|$.

Let $\varepsilon > 0$ small enough s.t $d = c + \varepsilon < 1$



For that ε, there is n_0 s.t for $n \geq n_0$:

$$\left|\frac{x_{n+1}}{x_n}\right| - \underbrace{|c|}_{= c \geq 0} \leq \left|\left|\frac{x_{n+1}}{x_n}\right| - c\right| < \varepsilon$$

$$\Rightarrow \left|\frac{x_{n+1}}{x_n}\right| < c + \varepsilon = d.$$

$$\left.\begin{array}{l} |x_n| \leq d\,|x_{n-1}| \\ \vdots \\ |x_{n_0+1}| < d\,|x_{n_0}| \end{array}\right\} |x_n| < d^{\,n-n_0}\,|x_{n_0}|$$

ii. **(2 points)** Deduce that x_n converges and find its limit.

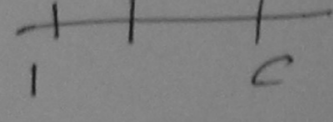
Same as (a) ii.

(c) Finally, assume that $c > 1$ and that $\left|\frac{x_{n+1}}{x_n}\right|$ converges to c.

i. **(3 points)** Show that $\{|x_n|\}$ diverges to $+\infty$.

$d = c - \varepsilon$

Let $\varepsilon > 0$ small enough st



$d = c - \varepsilon > 1$

For that ε, there is n_0 st for $n \geq n_0$

$$-\left|\frac{x_{n+1}}{x_n}\right| + |c| \leq \left|\left|\frac{x_{n+1}}{x_n}\right| - c\right| < \varepsilon$$

$$\Rightarrow \left|\frac{x_{n+1}}{x_n}\right| > c - \varepsilon = d$$

$$\left.\begin{array}{l} |x_n| > d\,|x_{n-1}| \\ \vdots \\ |x_{n_0+1}| > d\,|x_{n_0}| \end{array}\right\} |x_n| > d^{n-n_0}|x_{n_0}|$$

$\downarrow$

$+\infty$ since $d > 1$

$$\Rightarrow |x_n| \longrightarrow +\infty.$$

ii. **(5 points)** Does it imply that $\{x_n\}$ diverges to either $+\infty$ or $-\infty$

No for instance $x_n = (-1)^n \cancel{n^2}\, 2^n$

$\{x_n\}$ satisfies:

$$\left\{\begin{array}{l} \left|\frac{x_{n+1}}{x_n}\right| = \cancel{\frac{(n+1)^2}{n^2}} \; \frac{2^{n+1}}{2^n} = 2 \\ \text{But } x_n \not\to +\infty \text{ and } x_n \not\to -\infty. \end{array}\right.$$